TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 336, Number 2, April 1993

ON LAGRANGE INTERPOLATION AT DISTURBED ROOTS OF UNITY

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ABSTRACT. Let $z_{nk} = e^{it_{nk}}$, $0 \leq t_{n0} < \cdots < t_{nn} < 2\pi$, f a function in the disc algebra A, and $\mu_n = \max\{|t_{nk} - 2k\pi/(n+1)|: 0 \leq k \leq n\}$. Denote by $L_n(f;\cdot)$ the polynomial of degree n that agrees with f at $\{z_{nk}: k = 0, \ldots, n\}$. In this paper, we prove that for every p, $0 < p < \infty$, there exists a $\delta_p > 0$, such that $\|L_n(f;\cdot) - f\|_p = O(\omega(f; \frac{1}{n}))$ whenever $\mu_n \leq \delta_p/(n+1)$. It must be emphasized that δ_p necessarily depends on p, in the sense that there exists a family $\{z_{nk}: k = 0, \ldots, n\}$ with $\mu_n = \delta_2/(n+1)$ and such that $\|L_n(f;\cdot) - f\|_2 = O(\omega(f; \frac{1}{n}))$ for all $f \in A$, but $\sup\{\|L_n(f;\cdot)\|_p: f \in A, \|f\|_\infty = 1\}$ diverges for sufficiently large values of p. In establishing our estimates, we also derive a Marcinkiewicz-Zygmund type inequality for $\{z_{nk}\}$.

1. INTRODUCTION

Let D be the open unit disc in the complex plane with closure $\overline{D}$ and boundary T. Also, let $z_{nk} = e^{it_{nk}}$, $0 \leq t_{n0} < \cdots < t_{nn} < 2\pi$, and for a function f defined on T, let $L_n(f;\cdot)$ be the Lagrange polynomial of degree n that interpolates f at $\{z_{nk}: k = 0, \ldots, n\}$. If f is analytic on $\overline{D}$, then the following result is well known (cf. [17, Chapter 7]).

Theorem A. *For any f analytic on $\overline{D}$, a necessary and sufficient condition for*

$$\|L_n(f;\cdot) - f\|_\infty \to 0$$

is that the family $\{z_{nk}: k = 0, \ldots, n\}$ is uniformly distributed on T.

Here and throughout, we use the usual notation:

$$\|f\|_p = \begin{cases} \left\{ \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \right\}^{\frac{1}{p}} & \text{for } 0 < p < \infty, \\ \displaystyle\sup_{|z|=1} |f(z)| & \text{for } p = \infty. \end{cases}$$

Received by the editors April 26, 1990 and, in revised form, January 10, 1991.

1991 *Mathematics Subject Classification.* Primary 41A05; Secondary 30A10, 41A10.

Key words and phrases. Disturbed roots of unity, Marcinkiewicz-Zygmund type inequality, Lagrange interpolation, order of approximation, A_p-weights, H^p-interpolation.

The first author was supported by NSF Grants DMS-89-01345 and INT-87-12424, and SDIO/IST managed by ARO under contract DAAL 03-90-G-0091.

The second author was supported by NSFC and the Chinese National Commission of Education.

In this paper, we consider Lagrange interpolation of functions in the disc algebra A; that is, functions which are analytic in D and continuous on $\overline{D}$. It is well known, however, that the above result does not hold for $f \in A$ in general (cf. [6, 16]). In fact, Vértesi [16] proved in 1982 that for *any* family $\{z_{nk}\colon k = 0, \dots, n\}$, there exists an $f_0 \in A$ such that

$$\limsup_{n\to\infty} |L_n(f_0\,;\cdot)| = \infty$$

almost everywhere on T. For this reason, we must consider convergence in $L_p = L_p(T)$, $0 < p < \infty$. In this direction, probably the earliest result is due to Lozinski [10] in 1941, as follows.

Theorem B. *For* $0 < p < \infty$ *and* $z_{nk} = \exp(i2k\pi/(n+1))$,

$$\|L_n(f\,;\cdot) - f\|_p \to 0$$

for all $f \in A$.

Related results on interpolation at the roots of unity have also been obtained by Walsh and Sharma [18], Sharma and Vértesi [13], Saff and Walsh [12], and Shen [14], and at Fejér points on a Jordan curve by Curtiss [4], Al′per and Kalinogorskaja [1], Shen and Zhong [15], and Chui and Shen [2]. In this paper, we consider sample points z_{nk} which are not necessarily the $(n+1)$th roots of unity.

In order to give a sharp estimate of the order of convergence, we first establish the following Marcinkiewicz-Zygmund type inequality which is of independent interest. To facilitate our presentation, we need the following notation:

$$\pi_n = \text{ the class of polynomials of degree at most } n$$

and

$$\mu_n = \max_{0\le k\le n} \left| t_{nk} - \frac{2k\pi}{n+1} \right|. \tag{1.1}$$

Theorem 1. *For any* p, $1 < p < \infty$, *there exist positive constants* δ_p *and* C_p, *such that whenever*

$$\mu_n \le \frac{\delta_p}{n+1}, \tag{1.2}$$

then

$$\frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{i\theta})|^p \, d\theta \le \frac{C_p}{n+1} \sum_{k=0}^{n} |P_n(z_{nk})|^p$$

for all $P_n \in \pi_n$, *where* $z_{nk} = e^{it_{nk}}$.

As usual, $E_n(f)$ will denote the error of uniform approximation of f on T from π_n, namely:

$$E_n(f) = \inf_{q\in\pi_n} \|f - q\|_\infty.$$

The main theorem in this paper is the following.

Theorem 2. *For any* p, $0 < p < \infty$, *there exist positive constants* δ_p *and* C'_p, *such that whenever* (1.2) *is satisfied, then*

$$\|L_n(f;\cdot) - f\|_p \le C'_p E_n(f)$$

for any $f \in A$.

We remark that for $1 < p < \infty$, the δ_p in the above two theorems are the same, and for $0 < p \le 1$, we may choose $\delta_p = \delta_{p'}$ for any $p' > 1$. As a simple consequence of the above theorem, we have the following result.

Corollary 1. *Under the hypotheses stated in Theorem* 2,

$$\|L_n(f;\cdot) - f\|_p \le C_{p,r} n^{-r} \omega\left(f^{(r)}; \frac{1}{n}\right)$$

for any $f \in A$ *with* $f(e^{it}) \in C^r[0, 2\pi]$, *where* $\omega(g;t)$ *is the uniform modulus of continuity of* g *and* $C_{p,r}$ *is an absolute constant, depending only on* p *and* r.

We remark that in condition (1.2) the constant δ_p must necessarily depend on p. In §4, we will construct a family $\{z_{nk}\colon k = 0, \ldots, n\}$ with $\mu_n \le \delta/(n+1)$, where $\delta > 0$, for which

$$\|L_n(f;\cdot) - f\|_2 \le C_2 E_n(f)$$

for all $f \in A$, but

$$\sup\{\|L_n(f;\cdot)\|_p\colon f \in A,\ \|f\|_\infty = 1\} \to \infty.$$

We also remark that a necessary condition for $\|L_n(f;\cdot) - f\|_p \to 0$, where $f \in A$ and $0 < p < \infty$, is that the family $\{z_{nk}\colon k = 0, \ldots, n\}$ is uniformly distributed on T. This will be proved in §5. (Recall from Theorem A that this is also a necessary condition for $\|L_n(f;\cdot) - f\|_\infty \to 0$.)

2. Preliminary results

We first derive a result in harmonic analysis which is perhaps of independent interest. Let $\varphi \in L^1(T)$. Recall that $\varphi \in \mathrm{BMO}$ with norm $\|\varphi\|_*$ if

$$\|\varphi\|_* := \sup_I \frac{1}{|I|} \int_I |\varphi(e^{i\theta}) - \varphi_I|\, d\theta < \infty,$$

where the supremum is taken over all arcs I on T with length $|I|$ and

$$\varphi_I = \frac{1}{|I|} \int_I \varphi(e^{i\theta})\, d\theta.$$

The following is a well-known result due to John-Nirenberg (cf. [8, Chapter 6]).

Theorem C. *There exist positive constants* m *and* M *such that for any* $\varphi \in$ *BMO, any arc* $I \subset T$, *and any* $\lambda > 0$,

$$\frac{|\{e_{i\theta} \in I\colon |\varphi(e^{i\theta}) - \varphi_I| > \lambda\}|}{|I|} \le M \exp\left\{\frac{-m\lambda}{\|\varphi\|_*}\right\}.$$

We remark that the constants m and M can be chosen to be $m = \frac{1}{4e}$ and $M = \sqrt{e}$.

The following result on the BMO norm can be found in [9].

Theorem D. *There exists an absolute constant* C_* *such that for any* $g(e^{i\theta}) \in BMO$ *with* $g(z)$ *analytic in* $|z| > 1$ *and bounded at* ∞,

$$\|g(e^{i\theta})\|_* \le C_* \inf\{\|g - h\|_\infty \colon h \in H^\infty\}.$$

For any $\delta > 0$ and $1 < p < \infty$, a nonnegative function w defined on T is called an A_p-weight relative to δ if

$$\sup_I \left(\frac{1}{|I|}\int_I w(e^{i\theta})\,d\theta\right)\left(\frac{1}{|I|}\int_I (w(e^{i\theta}))^{-\frac{1}{p-1}}\,d\theta\right)^{p-1} < \delta. \tag{2.1}$$

For A_p-weights, the following result is due to Muckenhoupt [11].

Theorem E. *Let* $1 < p < \infty$ *and* $w(e^{i\theta})$ *be an* A_p*-weight relative to some* $\delta > 0$. *Then for any* $g \in L^p(T)$, *its Cauchy transform*

$$(Hg)(z) := \frac{1}{2\pi i}\int_T \frac{g(\zeta)}{\zeta - z}\,d\zeta, \qquad |z| < 1, \tag{2.2}$$

satisfies

$$\int_0^{2\pi} |(Hg)(e^{i\theta})|^p w(e^{i\theta})\,d\theta \le C_H \int_0^{2\pi} |g(e^{i\theta})|^p w(e^{i\theta})\,d\theta, \tag{2.3}$$

where C_H *is an absolute constant depending only on* δ *and* p.

Of course, $(Hg)(e^{i\theta})$ in inequality (2.3) is the almost everywhere radial limit of $(Hg)(re^{i\theta})$.

We have the following result.

Lemma 1. *For any* p, $1 < p < \infty$, *there exists an* $\varepsilon_p > 0$, *such that for all* $\varphi \in BMO$ *with* $\|\varphi\|_* < \varepsilon_p$, $|e^{\varphi}|$ *is an* A_p*-weight relative to* $\delta = 2$.

Proof. Since $\|\operatorname{Re}\varphi\|_* \le \|\varphi\|_*$, we may assume, without loss of generality, that φ is a real-valued function. Set $\psi = -\varphi/(p-1)$ and $w = e^{\varphi}$. Then we have

$$\begin{aligned}&\left(\frac{1}{|I|}\int_I w\,d\theta\right)\left(\frac{1}{|I|}\int_I w^{-\frac{1}{p-1}}\,d\theta\right)^{p-1} \\ &\quad = \left(\frac{1}{|I|}\int_I e^{\varphi-\varphi_I}\,d\theta\right)\left(\frac{1}{|I|}\int_I e^{\psi-\psi_I}\,d\theta\right)^{p-1},\end{aligned} \tag{2.4}$$

where it follows from Theorem C that

$$\begin{aligned}\frac{1}{|I|}\int_I e^{\varphi-\varphi_I}\,d\theta &= \frac{1}{|I|}\int_0^\infty |\{z \in I\colon e^{\varphi(z)-\varphi_I} > \lambda\}|\,d\lambda \\ &\le \frac{1}{|I|}\int_0^\infty |\{z \in I\colon |\varphi(z) - \varphi_I| > \ln\lambda\}|\,d\lambda \\ &\le 1 + \frac{1}{|I|}\int_1^\infty |\{z \in I\colon |\varphi(z) - \varphi_I| > \ln\lambda\}|\,d\lambda \\ &\le 1 + \int_1^\infty M\exp\left\{-\frac{m\ln\lambda}{\|\varphi\|_*}\right\}\,d\lambda = 1 + \frac{M\|\varphi\|_*}{m - \|\varphi\|_*}.\end{aligned}$$

Similarly, we also have

$$\frac{1}{|I|}\int_I e^{\psi-\psi_I}\,d\theta \le 1 + \frac{M\|\psi\|_*}{m - \|\psi\|_*} = 1 + \frac{M\|\varphi\|_*}{(p-1)m - \|\varphi\|_*}.$$

Combining these two estimates in (2.4), we have

$$\begin{aligned}(2.5)\qquad &\left(\frac{1}{|I|}\int_I w\,d\theta\right)\left(\frac{1}{|I|}\int_I w^{-\frac{1}{p-1}}\,d\theta\right)^{p-1}\\ &\le\left(1+\frac{M\|\varphi\|_*}{m-\|\varphi\|_*}\right)\left(1+\frac{M\|\varphi\|_*}{(p-1)m-\|\varphi\|_*}\right)^{p-1}<2\end{aligned}$$

for all $\varphi\in\mathrm{BMO}$ with sufficiently small $\|\varphi\|_*$. □

Next, to go from T to D, we introduce the function

$$(2.6)\qquad \omega_n(z)=\prod_{k=0}^{n}\left(z-\left(1-\frac{1}{n+1}\right)z_{nk}\right),$$

where $z_{nk}=e^{it_{nk}}$. Our key lemma for establishing Theorem 1 is the following.

Lemma 2. *Let $1<p<\infty$ and μ_n be defined as in (1.1). Then there exists a positive constant δ_p such that whenever $\mu_n\le\delta_p/(n+1)$, $|\omega_n(e^{i\theta})|^p$ is an A_p-weight relative to 2^{p+1}.*

Proof. We introduce three more polynomials:

$$(2.7)\qquad \begin{cases}\omega_n^*(z)=\displaystyle\prod_{k=0}^{n}\left(z-\left(1+\frac{1}{n+1}\right)z_{nk}\right),\\ \tilde\omega_n(z)=z^{n+1}-\left(1-\dfrac{1}{n+1}\right)^{n+1},\\ \tilde\omega_n^*(z)=z^{n+1}-\left(1+\dfrac{1}{n+1}\right)^{n+1}.\end{cases}$$

Clearly, $\ln\left(\frac{\omega_n(z)}{\tilde\omega_n(z)}\right)$ is in BMO on T, analytic in $|z|>1$, and bounded at ∞, so that it follows from Theorem D that

$$\begin{aligned}&\left\|\ln\left(\frac{\omega_n(e^{i\theta})}{\tilde\omega_n(e^{i\theta})}\right)\right\|_*\le C_*\left\|\ln\left(\frac{\omega_n(z)}{\tilde\omega_n(z)}\right)-\ln\left(\frac{\omega_n^*(z)}{\tilde\omega_n^*(z)}\right)\right\|_\infty\\ (2.8)\qquad &=C_*\max_\theta\left|\sum_{k=0}^{n}\ln\frac{(e^{i\theta}-(1-\frac{1}{n+1})e^{it_{nk}})(e^{i\theta}-(1+\frac{1}{n+1})e^{i\frac{2k\pi}{n+1}})}{(e^{i\theta}-(1+\frac{1}{n+1})e^{it_{nk}})(e^{i\theta}-(1-\frac{1}{n+1})e^{i\frac{2k\pi}{n+1}})}\right|\\ &=C_*\max_\theta\left|\sum_{k=0}^{n}\ln\left(1+\frac{2e^{i\theta}(e^{it_{nk}}-e^{i\frac{2k\pi}{n+1}})}{(n+1)(e^{i\theta}-(1+\frac{1}{n+1})e^{it_{nk}})(e^{i\theta}-(1-\frac{1}{n+1})e^{i\frac{2k\pi}{n+1}})}\right)\right|.\end{aligned}$$

To estimate the quantity

$$(2.9)\qquad I_n(\theta):=\frac{2e^{i\theta}(e^{it_{nk}}-e^{i\frac{2k\pi}{n+1}})}{(n+1)\left(e^{i\theta}-\left(1+\frac{1}{n+1}\right)e^{it_{nk}}\right)\left(e^{i\theta}-\left(1-\frac{1}{n+1}\right)e^{i\frac{2k\pi}{n+1}}\right)}$$

on $[0,2\pi]$, it is sufficient to consider $0\le\theta\le\frac{\pi}{n+1}$. We separate the estimation of the denominator of $I_n(\theta)$ in (2.9) into two cases:

(i) For $k = 0$ and $0 \le \theta \le \frac{\pi}{n+1}$, we have

$$\left|e^{i\theta} - \left(1 + \frac{1}{n+1}\right)e^{it_{nk}}\right| \ge \left(1 + \frac{1}{n+1}\right) - 1 = \frac{1}{n+1}.$$

(ii) For $k \ge 1$ and $0 \le \theta \le \frac{\pi}{n+1}$, we assume, without loss of generality, that $\delta_p \le \frac{1}{30}$, so that

$$\left|t_{nk} - \frac{2k\pi}{n+1}\right| \le \frac{1}{30(n+1)}, \qquad k = 1, \dots, n.$$

Hence, it follows that, for $k \ge 1$,

$$\begin{aligned}\left|e^{i\theta} - \left(1 + \frac{1}{n+1}\right)e^{it_{nk}}\right| &\ge |e^{i\theta} - e^{it_{nk}}| - \frac{1}{n+1} \ge \frac{2}{\pi}|\theta - t_{nk}| - \frac{1}{n+1}\\ &\ge \frac{2}{\pi}\left(\left|\theta - \frac{2k\pi}{n+1}\right| - \left|\frac{2k\pi}{n+1} - t_{nk}\right|\right) - \frac{1}{n+1}\\ &\ge \frac{2}{\pi}\left(\frac{(2k-1)\pi}{n+1} - \frac{1}{30(n+1)}\right) - \frac{1}{n+1}\\ &\ge \frac{4k}{3(n+1)}.\end{aligned}$$

Combining the estimates in (i) and (ii), we have

$$\left|e^{i\theta} - \left(1 + \frac{1}{n+1}\right)e^{it_{nk}}\right| \ge \frac{2(k+1)}{3(n+1)}, \qquad k = 0, \dots, n.$$

The same lower bound also applies to the quantity

$$\left|e^{i\theta} - \left(1 - \frac{1}{n+1}\right)e^{i\frac{2k\pi}{n+1}}\right|.$$

Consequently, we have the following estimate of $I_n(\theta)$ defined in (2.9):

$$\begin{aligned}|I_n(\theta)| &\le \frac{9}{2}\cdot\frac{n+1}{(k+1)^2}|e^{it_{nk}} - e^{i\frac{2k\pi}{n+1}}|\\ &\le \frac{9}{2}\cdot\frac{n+1}{(k+1)^2}\left|t_{nk} - \frac{2k\pi}{n+1}\right| \le \frac{9}{2}\cdot\frac{n+1}{(k+1)^2}\mu_n.\end{aligned}$$

Recalling that $\delta_p \le \frac{1}{30}$, we have, for $\mu_n \le \frac{\delta_p}{n+1}$,

$$|I_n(\theta)| \le \frac{9}{2}\frac{n+1}{(k+1)^2}\mu_n \le \frac{3}{20} < \frac{2}{3}, \qquad k = 0, \dots, n. \tag{2.10}$$

Note that for $|\zeta| \le \frac{2}{3}$, we have $|\ln(1+\zeta)| \le 2|\zeta|$, so that by using $\zeta = I_n(\theta)$ and estimate (2.10), the result in (2.8) yields

$$\left\|\ln\left(\frac{\omega_n(e^{i\theta})}{\tilde{\omega}_n(e^{i\theta})}\right)\right\|_* \le 9C_*(n+1)\mu_n \sum_{K=0}^{n} \frac{1}{(k+1)^2} < 18C_*(n+1)\mu_n.$$

Hence, it follows from Lemma 1 that if

$$\mu_n \leq \frac{\varepsilon_p}{18C_* p(n+1)}, \tag{2.11}$$

then $\left|\frac{\omega_n(e^{i\theta})}{\tilde{\omega}_n(e^{i\theta})}\right|^p$ is an A_p-weight relative to $\delta = 2$. However, it is clear that since

$$2^{-p} \leq |\tilde{\omega}_n(e^{i\theta})|^p \leq 2^p,$$

$|\omega_n(e^{i\theta})|^p$ is also an A_p-weight relative to $\delta = 2^{p+1}$. In view of (2.11), this completes the proof of Lemma 2 by choosing

$$\delta_p = \min\left(\frac{\varepsilon_p}{18C_* p}, \frac{1}{30}\right). \quad \square \tag{2.12}$$

In what follows, we need a result on H^p-interpolation. As usual, a sequence $\{\zeta_j\}$, $j = 1, 2, \ldots$, in D is said to be *δ-uniformly separated*, where $\delta > 0$, if

$$\prod_{\substack{j=1 \\ j \neq k}}^{\infty} \left|\frac{\zeta_j - \zeta_k}{1 - \overline{\zeta}_k \zeta_j}\right| \geq \delta > 0 \quad \text{all } k. \tag{2.13}$$

The following result can be found in [5, Chapter 9, p. 149].

Theorem F. *Let $0 < p < \infty$ and $\{\zeta_j\}$ be a δ-uniformly separated sequence in D. Then for any sequence of complex numbers $\{a_j\}$ satisfying*

$$\sum_{j=1}^{\infty} |a_j|^p (1 - |\zeta_j|^2) < \infty,$$

there exists a function $g \in H^p$, such that
(i) *$g(\zeta_j) = a_j$, $j = 1, 2, \ldots$, and*
(ii)

$$\frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})|^p \, d\theta \leq C_{p,\delta} \sum_{j=1}^{\infty} |a_j|^p (1 - |\zeta_j|^2)$$

where $C_{p,\delta}$ is an absolute constant depending only on p and δ.

We remark that the above theorem holds for any finite sequence, provided that δ is independent of the length of this sequence. In order to apply the above theorem, we need the following.

Lemma 3. *Let μ_n satisfy* (1.2). *Then the sequence $\{(1 - \frac{1}{n+1})e^{it_{nk}}\}$, $k = 0, \ldots, n$, is δ-uniformly separated for some $\delta > 0$ independent of n.*

Proof. To simplify the notation, set

$$\rho = \left(1 - \frac{1}{n+1}\right) \quad \text{and} \quad \zeta_k = \rho z_{nk} = \left(1 - \frac{1}{n+1}\right) e^{it_{nk}},$$

where $z_{nk} = e^{it_{nk}}$. Then

$$\begin{aligned} \left|\frac{\zeta_j - \zeta_k}{1 - \overline{\zeta}_k \zeta_j}\right|^2 &= \frac{2\rho^2(1 - \cos(t_{nj} - t_{nk}))}{1 - 2\rho^2 \cos(t_{nj} - t_{nk}) + \rho^4} \\ &= \frac{4\rho^2 \sin^2\left(\frac{t_{nj} - t_{nk}}{2}\right)}{(1 - \rho^2)^2 + 4\rho^2 \sin^2\left(\frac{t_{nj} - t_{nk}}{2}\right)}. \end{aligned} \tag{2.14}$$

On the other hand, from the hypothesis, we have

$$
\begin{aligned}
|t_{nj} - t_{nk}| &\geq \left|\frac{2j\pi}{n+1} - \frac{2k\pi}{n+1}\right| - \left|t_{nj} - \frac{2j\pi}{n+1}\right| - \left|t_{nk} - \frac{2k\pi}{n+1}\right| \\
&\geq \frac{2\pi|j-k|}{n+1} - \frac{2\delta_p}{n+1} \qquad (2.15)\\
&\geq \frac{2\pi}{n+1}\left(|j-k| - \frac{1}{4}\right) > \frac{\pi}{n+1}|j-k|
\end{aligned}
$$

for $j \neq k$, by recalling from (2.12) that $\delta_p \leq \frac{1}{30} < \frac{\pi}{4}$. Hence, since $\sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \pi/2$ and, as a function of x,

$$\frac{4\rho^2 x}{(1-\rho^2)^2 + 4\rho^2 x}$$

is monotonically increasing for $x \geq 0$, we have for $0 \leq t_{nj} - t_{nk} \leq \pi$ or $-\pi \leq t_{nj} - t_{nk} \leq 0$, by applying (2.15):

$$\left|\frac{\zeta_j - \zeta_k}{1 - \overline{\zeta}_k \zeta_j}\right|^2 \geq \frac{4\rho^2\left(\frac{2}{\pi}\right)^2\left(\frac{\pi}{2(n+1)}\right)^2(j-k)^2}{(1-\rho^2)^2 + 4\rho^2\left(\frac{2}{\pi}\right)^2\left(\frac{\pi}{2(n+1)}\right)^2(j-k)^2}.$$

Consequently, in view of the fact that each term is less than 1, we may conclude from (2.14) that

$$\prod_{\substack{j=0\\ j\neq k}}^{n}\left|\frac{\zeta_j - \zeta_k}{1 - \overline{\zeta}_k \zeta_j}\right|^2 \geq \prod_{l=1}^{n}\left(\frac{4\rho^2\left(\frac{2}{\pi}\right)^2\left(\frac{\pi}{2(n+1)}\right)^2 l^2}{(1-\rho^2)^2 + 4\rho^2\left(\frac{2}{\pi}\right)^2\left(\frac{\pi}{2(n+1)}\right)^2 l^2}\right)^2. \qquad (2.16)$$

To estimate the lower bound in (2.16), we recall that $\rho = (1 - \frac{1}{n+1})$, so that

$$(1-\rho^2)^2 \leq \frac{4}{(n+1)^2} \leq \frac{8\rho^2}{(n+1)^2},$$

and (2.16) yields

$$\prod_{\substack{j=0\\ j\neq k}}^{n}\left|\frac{\zeta_j - \zeta_k}{1 - \overline{\zeta}_k \zeta_j}\right| \geq \prod_{l=1}^{n}\frac{l^2}{2+l^2} \geq \prod_{l=1}^{\infty}\left(1 - \frac{2}{2+l^2}\right) =: \delta,$$

where $\delta > 0$, since $\sum_{l=1}^{\infty} 2/(2+l^2) < \infty$. □

3. Proof of the theorems

We are now ready to prove Theorem 1. For any polynomial $P_n \in \pi_n$, we use the notation

$$P_n^*(z) = P_n\left(\left(1 - \frac{1}{n+1}\right)^{-1} z\right), \qquad (3.1)$$

so that

$$P_n^*\left(\left(1 - \frac{1}{n+1}\right)e^{it_{nk}}\right) = P_n(z_{nk}), \qquad (3.2)$$

where $z_{nk} = e^{it_{nk}}$, and

$$\begin{aligned}\int_{|z|=1} |P_n(z)|^p |dz| &\le \int_{|z|=1+\frac{1}{n}} |P_n(z)|^p |dz| \\ &= \left(1+\frac{1}{n}\right) \int_{|z|=1} |P_n^*(z)|^p |dz|.\end{aligned} \tag{3.3}$$

From hypothesis (1.2) on μ_n, we can apply Lemma 3 and conclude from Theorem F that there exists a function $g \in H^p$ which satisfies

$$g\left(\left(1-\frac{1}{n+1}\right)e^{it_{nk}}\right) = P_n^*\left(\left(1-\frac{1}{n+1}\right)e^{it_{nk}}\right) = P_n(z_{nk}), \tag{3.4}$$

$k = 0, \ldots, n$, by using (3.2), and

$$\frac{1}{2\pi}\int_0^{2\pi} |g(e^{i\theta})|^p \, d\theta \le \frac{2C_{p,\delta}}{n+1} \sum_{k=0}^{n} |P_n(z_{nk})|^p, \tag{3.5}$$

by observing that

$$1 - \left|\left(1-\frac{1}{n+1}\right)e^{it_{nk}}\right|^2 \le \frac{2}{n+1}.$$

Hence, to complete the proof of Theorem 2, it is sufficient to show that

$$\int_0^{2\pi} |P_n(e^{i\theta})|^p \, d\theta \le C \int_0^{2\pi} |g(e^{i\theta})|^p \, d\theta \tag{3.6}$$

for some absolute constant C. Now, from (3.4), it is well known (cf. [17, Chapter 3]) that

$$g(z) - P_n^*(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\omega_n(z)}{\omega_n(\zeta)} \frac{g(\zeta)}{\zeta - z} \, d\zeta \tag{3.7}$$

for $|z| < 1$, where ω_n is defined in (2.6). By using the notation in (2.2), we have

$$\left(\frac{g - P_n^*}{\omega_n}\right)(z) = \left(H\left(\frac{g}{\omega_n}\right)\right)(z), \qquad |z| < 1.$$

Also, for $1 < p < \infty$ and under the condition (1.2), Lemma 2 allows us to conclude that $|\omega_n(e^{i\theta})|^p$ is an A_p-weight. Hence, from Theorem E, it follows that

$$\begin{aligned}\int_0^{2\pi} |g(e^{i\theta}) - P_n^*(e^{i\theta})|^p \, d\theta &= \int_0^{2\pi} \left|\left(H\left(\frac{g}{\omega_n}\right)\right)(e^{i\theta})\right|^p |\omega_n(e^{i\theta})|^p \, d\theta \\ &\le C_H \int_0^{2\pi} |g(e^{i\theta})|^p \, d\theta,\end{aligned}$$

so that, by Minkowski's inequality,

$$\int_0^{2\pi} |P_n^*(e^{i\theta})|^p \, d\theta \le \left(1 + C_H^{\frac{1}{p}}\right)^p \int_0^{2\pi} |g(e^{i\theta})|^p \, d\theta. \tag{3.8}$$

Finally, by applying (3.3) and (3.8), we arrive at (3.6) with $C \le 2(1 + C_H^{\frac{1}{p}})^p$. This completes the proof of Theorem 1. □

To prove Theorem 2, we first remark that the case $0 < p \leq 1$ can be reduced to the case $1 < p < \infty$ by a simple application of Hölder's inequality (cf. [15]). Hence, we now assume that $1 < p < \infty$ and that (1.2) is satisfied. Let $f \in A$ and $\widehat{P}_n$ be its best uniform approximant from π_n; that is, $\widehat{P}_n \in \pi_n$ and

$$\|\widehat{P}_n - f\|_\infty = E_n(f). \tag{3.9}$$

Since $L_n(\widehat{P}_n; \cdot) = \widehat{P}_n$, we have

$$\|L_n(f; \cdot) - f\|_p \leq \|\widehat{P}_n - f\|_p + \|L_n(f - \widehat{P}_n; \cdot)\|_p. \tag{3.10}$$

Also, since $L_n(f - \widehat{P}_n; z_{nk}) = f(z_{nk}) - \widehat{P}_n(z_{nk})$, it follows from Theorem 1 that

$$\|L_n(f - \widehat{P}_n; \cdot)\|_p^p \leq \frac{C_p}{n+1} \sum_{k=0}^{n} |f(z_{nk}) - \widehat{P}_n(z_{nk})|^p.$$

Hence, an application of (3.9) and (3.10) yields:

$$\|L_n(f; \cdot) - f\|_p \leq \left(1 + C_p^{\frac{1}{p}}\right) E_n(f).$$

This completes the proof of Theorem 2 with $C_p' = 1 + C_p^{\frac{1}{p}}$. □

4. Dependence of δ_p on p

In this section, we construct a family $\{z_{nk}\colon k = 0, \ldots, n\}$ on T with $\mu_n \leq \frac{\delta_2}{n+1}$ for some constant $\delta_2 > 0$ such that

$$\|L_n(f; \cdot) - f\|_2 \leq C_2' E_n(f) \tag{4.1}$$

for all $f \in A$, but

$$\sup\{\|L_n(f; \cdot)\|_p \colon f \in A, \|f\|_\infty = 1\} \to \infty \tag{4.2}$$

for all sufficiently large values of p.

According to Theorem 1, there exists a δ_2 with $\frac{2\pi}{n+1} > \delta_2 > 0$ such that if we select

$$z_{nk} = \begin{cases} \exp\left(i\frac{2k\pi + \delta_2}{n+1}\right) & \text{for } 0 \leq k \leq [\frac{n}{2}], \\ \exp\left(i\frac{2k\pi}{n+1}\right) & \text{for } [\frac{n}{2}] < k \leq n, \end{cases} \tag{4.3}$$

then (4.1) is satisfied. Let $\lambda_n(z) = \prod_{k=0}^{n}(z - z_{nk})$. Then we may also write

$$\lambda_n(z) = (z^{n+1} - 1) \prod_{k=0}^{[\frac{n}{2}]} \frac{z - z_{nk}}{z - e^{i\frac{2k\pi}{n+1}}}. \tag{4.4}$$

Hence, for $\frac{5}{8}n \le j \le \frac{7}{8}n$, we have

$$\begin{aligned}|\lambda_n'(z_{nj})| &= (n+1)\left| z_{nj}^n \prod_{k=0}^{[\frac{n}{2}]} \frac{z_{nj}-z_{nk}}{z_{nj}-e^{i\frac{2k\pi}{n+1}}}\right| \\ &\le (n+1)\prod_{k=0}^{[\frac{n}{2}]}\left(1+\left|\frac{1-e^{i\frac{\delta_2}{n+1}}}{1-e^{i\frac{2(k-j)\pi}{n+1}}}\right|\right) \\ &\le (n+1)\left(1+\frac{\frac{\delta_2}{n+1}}{2\sin\frac{\pi}{8}}\right)^{[\frac{n}{2}]+1} \le c_1 n\end{aligned}$$

for some constant $c_1 > 0$. This gives

$$\sum_{k=0}^{n}\frac{1}{|\lambda_n'(z_{nk})|} \ge \sum_{\frac{5n}{8}\le k\le \frac{7n}{8}} \frac{1}{|\lambda_n'(z_{nk})|} \ge \frac{1}{4c_1}. \tag{4.5}$$

On the other hand, for $\zeta_0 = e^{-i\frac{\pi}{2(n+1)}}$, we have, from (4.4),

$$\left|\frac{-i-1}{\lambda_n(\zeta_0)}\right| = \left|\frac{\zeta_0^{n+1}-1}{\lambda_n(\zeta_0)}\right| = \prod_{k=0}^{[\frac{n}{2}]}\left|1-\frac{e^{i\frac{2k\pi}{n+1}}-z_{nk}}{\zeta_0-z_{nk}}\right|. \tag{4.6}$$

It is obvious that there exists a constant $\varepsilon > 0$ such that whenever $0 \le k \le \varepsilon n$, we have

$$\left|\arg\frac{e^{i\frac{2k\pi}{n+1}}-z_{nk}}{\zeta_0-z_{nk}}\right| \le \frac{\pi}{4}.$$

Hence, it follows that

$$\begin{aligned}\prod_{0\le k\le \varepsilon n}\left|1-\frac{e^{i\frac{2k\pi}{n+1}}-z_{nk}}{\zeta_0-z_{nk}}\right| &\le \prod_{0\le k\le\varepsilon n}\left|1-\left|\frac{e^{i\frac{2k\pi}{n+1}}-z_{nk}}{\zeta_0-z_{nk}}\right|e^{i\frac{\pi}{4}}\right| \\ &\le c_2\prod_{1\le k\le \varepsilon n}\left(1-\frac{c_3\delta_2}{k}\right) \le c_2(\varepsilon n)^{-c_4\delta_2}\end{aligned} \tag{4.7}$$

for some absolute positive constants c_2, c_3, and c_4. In addition,

$$\begin{aligned}\prod_{\varepsilon n<k<[\frac{n}{2}]}\left|1-\frac{e^{i\frac{2k\pi}{n+1}}-z_{nk}}{\zeta_0-z_{nk}}\right| &\le \prod_{\varepsilon n<k\le[\frac{n}{2}]}\left(1+\left|\frac{e^{i\frac{2k\pi}{n+1}}-z_{nk}}{\zeta_0-z_{nk}}\right|\right) \\ &\le \prod_{\varepsilon n<k\le[\frac{n}{2}]}\left(1+\frac{c_5\delta_2}{k}\right) \le c_6\end{aligned} \tag{4.8}$$

for some absolute constants c_5 and c_6. By combining the information from (4.6), (4.7), and (4.8), we obtain

$$|\lambda_n(\zeta_0)| \ge c_7 n^{c_4\delta_2} \tag{4.9}$$

for some $c_7 > 0$. Hence, using (4.5) and (4.9), we arrive at

$$\sum_{k=0}^{n}\left|\frac{\lambda_n(\zeta_0)}{(\zeta_0-z_{nk})\lambda_n'(z_{nk})}\right| \ge \frac{c_7}{8c_1}n^{c_4\delta_2}. \tag{4.10}$$

To get rid of the absolute value, let

$$\arg \frac{\lambda_n(\zeta_0)}{(\zeta_0 - z_{nk})\lambda_n'(z_{nk})} = \theta_{nk}.$$

By a lemma in [3], there exists a function $f_n \in A$ satisfying $\|f_n\|_\infty = 1$ and $f_n(z_{nk}) = e^{-i\theta_{nk}}$, $k = 0, \dots, n$. Hence, (4.10) yields

$$|L_n(f_n; \zeta_0)| = \left| \sum_{k=0}^{n} f_n(z_{nk}) \frac{\lambda_n(\zeta_0)}{(\zeta_0 - z_{nk})\lambda_n'(z_{nk})} \right| \geq \frac{c_7}{8c_1} n^{c_4\delta_2}. \tag{4.11}$$

Finally, by one of the two Marcinkiewicz-Zygmunds inequalities (cf. [20, p. 30]) and (4.11), we have, for $p \geq 1$,

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} |L_n(f_n; e^{i\theta})|^p \, d\theta &\geq \frac{c_p''}{n+1} \sum_{k=0}^{n} |L_n(f_n; \zeta_0 e^{i\frac{2k\pi}{n+1}})|^p \\ &\geq \frac{c_p''}{n+1} |L_n(f_n; \zeta_0)|^p \geq c_8 n^{pc_4\delta_2 - 1} \end{aligned}$$

for some absolute constant c_8. Hence, using the fact that $\|f_n\|_\infty = 1$, we arrive at (4.2) for $p > \frac{1}{c_4\delta_2}$.

5. Final remarks

I. An assumption such as (1.2) on the distribution of $\{z_{nk}: k = 0, \dots, n\}$ on T is necessary for $\|L_n(f; \cdot) - f\|_p \to 0$ for all $f \in A$. We already know from Theorem A that for $p = \infty$, this family must necessarily be uniformly distributed on T. In the following, we will show that the uniform distribution of $\{z_{nk}: k = 0, \dots, n\}$ on T is also necessary for $\|L_n(f; \cdot) - f\|_p \to 0$ for all $f \in A$. Let

$$\eta_n(z) = \prod_{k=0}^{n} (z - z_{nk}).$$

It is well known (cf. [7, Chapter 2]) that the uniform distribution of the family $\{z_{nk}: k = 0, \dots, n\}$ on T is equivalent to

$$\lim_{n\to\infty} \frac{|\eta_n(z)|^{\frac{1}{n+1}}}{|z|} = 1, \qquad |z| > 1, \tag{5.1}$$

where the convergence is uniform on every compact subset in $|z| > 1$. Assume that (5.1) does not hold. Then it is known (cf. [17, p. 161]) that there is some ζ_0, $|\zeta_0| > 1$, and a sequence of integers $n_k \to \infty$ such that $|\eta_{n_k}(\zeta_0)|^{\frac{1}{n_k+1}} < 1$ for all k, so that

$$|\eta_{n_k}(\zeta_0)| < (1 - \varepsilon_0)^{n_k+1} \tag{5.2}$$

for some $\varepsilon_0 > 0$ and all k. Consider the function $f_0(z) = 1(\zeta_0 - z)$ which is in A. Then it follows from the formula

$$f_0(z) - L_n(f_0; z) = \frac{\eta_n(z)}{\eta_n(\zeta_0)(\zeta_0 - z)}$$

that

$$|f_0(0) - L_{n_k}(f_0; 0)| = \frac{1}{|\eta_{n_k}(\zeta_0)||\zeta_0|},$$

which tends to ∞ in view of (5.2). Hence, by [5, Theorem 1.5],

$$\|L_n(f_0; \cdot) - f_0\|_p \not\to 0.$$

II. Recall that two of the main tools in establishing Theorem 1, and hence Theorem 2, are the H^p-interpolation result stated in Theorem F and the integral representation formula in equality (3.7). These two results, however, can be generalized to multiple nodes z_{nk}. In addition, if each z_{nk}, $k = 0, \ldots, n$, has the same multiplicity, then Lemma 2 also applies, since $|\omega_n(e^{i\theta})|^\alpha$ is always an A_p-weight for any $\alpha > 0$. Hence, for any nonnegative integer q, by setting $N = (q+1)(n+1) - 1$, a simple modification of our proof of Theorem 1 yields the inequality

$$\frac{1}{2\pi}\int_0^{2\pi} |P_N(e^{i\theta})|^p\, d\theta \le \frac{C_p}{n+1}\sum_{k=0}^{n}\sum_{j=0}^{q}\frac{|P_N^{(j)}(z_{nk})|^p}{(n+1)^{jp}}$$

for all $P_N \in \pi_N$ provided that (1.2) is satisfied, where $1 < p < \infty$. Of course, a different value of δ_p is required. Consequently, Theorem 2 can be easily extended to Hermite or Hermite-Fejér interpolation with the same estimates. In [20], this result was established for the roots of unities using another method.

Of course, the analogous problems for nodes z_{nk} with different multiplicities still remain open.

III. A seemingly very difficult problem is to determine the largest δ_p in the condition (1.2) for the validity of Theorems 1 and 2.

IV. A more interesting problem is to find a necessary and sufficient condition on the distribution of $\{z_{nk}: k = 0, \ldots, n\}$ so that $\|L_n(f; \cdot) - f\|_p \to 0$ for all $f \in A$, where $0 < p < \infty$. Recall that (1.2) is a sufficient condition and the uniform distribution on T is a necessary condition. We remark that an example can be constructed to show that the uniform distribution of $\{z_{nk}: k = 0, \ldots, n\}$ on T is not sufficient for L_p convergence, $0 < p < \infty$.

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